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CONTACT OF ELASTIC BODIES WITH THIN VISCO-ELASTIC COATINGS UNDER CONDITIONS OF ROLLING OR SLIDING FRICTION†

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Analytic methods are developed for solving the contact problem of the rolling of an elastic cylinder along a visco-elastic layer bonded to an elastic base with the aim of studying the effect of the mechanical and geometric characteristics of their surface layers on the contact interaction parameters and the coefficient of rolling friction. A Maxwell model is used to describe the mechanical properties of the visco-elastic layer. The problem is treated assuming that there is partial siding in the contact area which makes it possible to treat the resistance to rolling as the overall result of the manifestation of the imperfect elasticity of the surface layers of the interacting bodies and sliding friction in the contact area. The solution of the problem of the total sliding of a cylinder on an elastic base covered with a thin visco-elastic layer is obtained as a special case.

The study of the effects of coatings and various surface films on the contact characteristics and resistance to motion accompanying the relative rolling or sliding of the interacting surfaces, when their properties of imperfect elasticity are clearly exhibited, is important in problems of increasing their useful lifetime and reducing friction and wear.

The stress–strain state of a visco-elastic strip bonded to a visco-elastic half-plane and subjected to the action of a moving load has been investigated in [1, 2] using the Fourier transform method. A numerical algorithm has been proposed [3] for solving a contact problem involving laminated elastic and visco-elastic bodies under conditions of rolling friction. However, it is difficult to use this algorithm when analysing contact characteristics and their dependence on all of the parameters of the problem. The dependence of the stressed state of a visco-elastic strip on time when a rigid cylindrical punch is embedded in it has been studied in [4].

1. FORMULATION OF THE PROBLEM

We will consider the contact problem in a planar formulation for an elastic cylinder and a base consisting of a visco-elastic strip of thickness h bonded to an elastic half-plane (Fig. 1). The cylinder rolls or slides along the base at a constant linear velocity V and angular velocity ω. The contacting surface of the cylinder is described by the function $f(x) = x^2/(2R)$ (R is the radius of the cylinder).

We introduce a fixed (x', y') system of coordinates connected with the base and a moving (x, y) system of coordinates connected with the moving cylinder. Here

$$x' = x + Vt, \quad y' = y$$

For uniform motion of the cylinder, the motion of the medium can be considered to be steady with respect to the system of coordinates (x, y). In this system of coordinates, the displacements and stresses do not depend on time explicitly and are functions of the coordinates (x, y).

Boundary conditions. Following Reynolds, we subdivide the contact area $(-a, b)$ into sliding zones (S) and adhesion zones (A). In the S zones, the sliding friction is modelled using the Coulomb–Amonton law

$$|\tau(x)| = \mu p(x) \text{ when } y = 0, \quad x \in S \tag{1.1}$$

where $\tau(x)$ and $p(x)$ are the shear and normal stresses on the contact area, respectively. In the A zones the tangential velocities of the contacting points of the cylinder and visco-elastic layer are equal. Hence,

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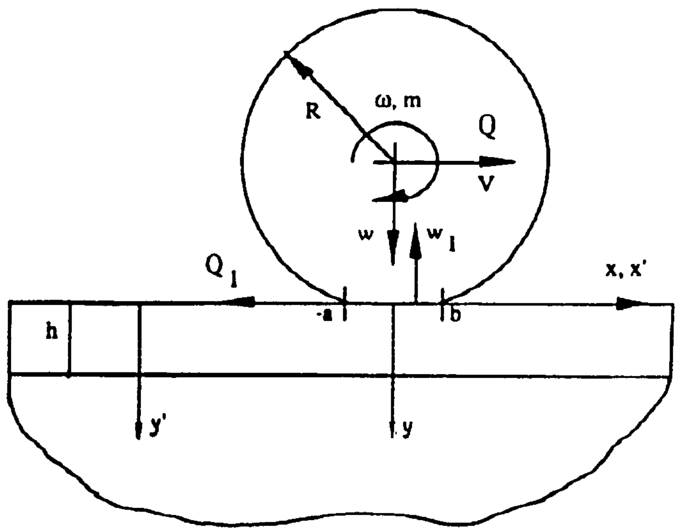

Fig. 1.

in the (x', y') system of coordinates, the tangential displacements u_1 and u of the cylinder and the base, respectively, satisfy the relation

$$\frac{du}{dt} = V - \omega R + \frac{du_1}{dt} \text{ when } y = 0, \quad x \in A \tag{1.2}$$

In the (x, y) system of coordinates, Eq. (1.2) has the form

$$-\frac{\partial u_1}{\partial x} + \frac{\partial u}{\partial x} = -\delta \text{ when } y = 0, \quad x \in A \quad \left(\delta = \frac{V - \omega R}{V}\right) \tag{1.3}$$

where δ is the magnitude of the relative sliding.

Furthermore, in the zones of adhesion A, the normal and shear stresses are related by the inequality

$$|\tau(x)| < \mu p(x) \tag{1.4}$$

Note that relation (1.1) holds over the whole of the contact area $(-a, b)$ in the case of complete sliding. It follows from the contact condition that the relation

$$\upsilon_1(x) + \upsilon_2(x) + \upsilon_3(x) = \gamma - x^2/(2R) \tag{1.5}$$

is satisfied for all points of the contact area $(-a, b)$. In (1.5), υ_1, υ_2 and υ_3 are the normal displacements of the boundary points of the cylinder, of the elastic half-plane and of the layer, respectively (which are assumed to be positive for each body), and γ is the penetration of the cylinder into the base.

It is assumed that the visco-elastic layer is bonded to the elastic half-plane and that the following boundary conditions holds at the interface ($y = h$)

$$\begin{aligned} u(x, h^-) &= u(x, h^+), \quad \upsilon(x, h^-) = \upsilon(x, h^+) \\ p(x, h^-) &= p(x, h^+), \quad \tau(x, h^-) = \tau(x, h^+) \end{aligned} \tag{1.6}$$

A mechanical model of the visco-elastic layer. Assuming that the thickness h of the visco-elastic layer is much less than the width of the contact area, we shall simulate its normal and tangential compliance using a one-dimensional Maxwellian model, namely

$$\dot{u}_3 = h\left(\frac{1}{\eta_\tau}\tau + \frac{1}{\lambda_\tau}\dot{\tau}\right), \quad \dot{\upsilon}_3 = h\left(\frac{1}{\eta_n}p + \frac{1}{\lambda_n}\dot{p}\right) \tag{1.7}$$

where u_3 and υ_3 are the tangential and normal displacements of the boundary of the layer ($y = 0$) and η_n, λ_n and η_τ, λ_τ are the visco-elastic characteristics of the layer in the normal and tangential directions, respectively. A similar rod model has been used previously in the case of an elastic base in [5].

In the (x, y) system of coordinates relations (1.7) have the form

$$\frac{\partial u_3}{\partial x} = -\frac{h}{V\eta_\tau}\tau(x) + \frac{h}{\lambda_\tau}\frac{d\tau(x)}{dx} \tag{1.8}$$

$$\frac{\partial \upsilon_3}{\partial x} = -\frac{h}{V\eta_n}p(x) + \frac{h}{\lambda_n}\frac{dp(x)}{dx} \tag{1.9}$$

In the model being considered it is assumed that the normal and shear stresses at the upper boundary of the layer ($y = 0$) and at the boundary between the layer and the elastic half plane ($y = h$) have the same values. The gradient of the displacements of the boundary of the elastic bodies (of the cylinder and the elastic half-plane) is therefore defined by the following relations ($i = 1, 2$)

$$\frac{\partial u_i}{\partial x} = -\frac{(1-2\nu_i)(1+\nu_i)}{E_i}p(x) - \frac{2(1-\nu_i^2)}{\pi E_i}\int_{-a}^{b}\frac{\tau(s)}{x-s}ds \tag{1.10}$$

$$\frac{\partial \upsilon_i}{\partial x} = \frac{(1-2\nu_i)(1+\nu_i)}{E_i}\tau(x) - \frac{2(1-\nu_i^2)}{\pi E_i}\int_{-a}^{b}\frac{p(s)}{x-s}ds \tag{1.11}$$

Equations (1.8)–(1.11), together with the boundary conditions (1.1), (1.3) and (1.5), are used to find the normal and shear stresses in the contact area $(-a, b)$.

2. ANALYSIS OF THE CONTACT PRESSURES

In order to simplify the calculations, we shall neglect the effect of the shear contact stresses on the normal contact stresses. Then, using relations (1.5), (1.9) and (1.11) (the latter is considered for $\tau(x) = 0$) and introducing the new variable ξ, which is related to x by the equality

$$x = \frac{b-a}{2} + \frac{a+b}{2}\xi$$

we obtain

$$\int_{-1}^{1}\frac{P(\sigma)d\sigma}{\xi-\sigma} + KP(\xi) - \frac{C}{D}P'(\xi) = D(L+\xi) \tag{2.1}$$

$$P(\xi) = \frac{2}{\pi E^*}p\left(\frac{b-a}{2} + \frac{a+b}{2}\xi\right), \quad K = \frac{h\pi E^*}{2V\eta_n}, \quad C = \frac{h\pi E^*}{2R\lambda_n}, \quad D = \frac{a+b}{2R}, \quad L = \frac{b-a}{a+b}$$

Bearing in mind the condition that the pressure at the ends of the contact area between the cylinder and the base is equal to zero, that is, $P(-1) = P(1) = (0)$, we transform Eq. (2.1) to a Fredholm equation of the second kind

$$\int_{-1}^{1} F(\sigma)\left[\ln|\xi-\sigma| + \frac{K}{2}\operatorname{sgn}(\xi-\sigma) - \frac{1+\sigma}{2}\ln(1+\sigma) - \frac{1-\sigma}{2}\ln(1-\sigma) + \frac{K\sigma}{2}\right]d\sigma - \frac{C}{D}F(\xi) = \xi D$$

$$(F(\xi) = P'(\xi)) \tag{2.2}$$

It follows from the condition of equilibrium of the normal forces applied to the cylinder that the required function $F(\xi)$ also satisfies the relation

$$W = -D\int_{-1}^{1} sF(s)ds \tag{2.3}$$

where $W = 2w/(\pi RE^*)$ is the dimensionless normal load applied to the cylinder.

Equations (2.2) and (2.3) are used to find the contact pressure $P(\xi)$, the size of the contact area D, the displacement of the contact area L and the maximum penetration of the cylinder into the visco-elastic layer Δ_{max}

$$
\begin{aligned}
&P(\xi)=\int_{-1}^{\xi} F(\sigma)d\sigma, \quad D=-W\left[\int_{-1}^{1} \rho F(\rho)d\rho\right]^{-1} \\
&L=\frac{1}{2D}\int_{-1}^{1} F(\sigma)[(1+\sigma)\ln(1+\sigma)+(1-\sigma)\ln(1-\sigma)-K\sigma]d\sigma \\
&\Delta_{max}=\max\frac{\upsilon_3(x)}{R}=D\max\Phi(\xi), \quad -1<\xi<1 \\
&\left(\Phi(\xi)=K\left[\int_{\xi}^{1}\tau F(\tau)d\tau+\xi\int_{-1}^{\xi}F(\tau)d\tau\right]+\frac{C}{D}\int_{\xi}^{1}F(\tau)d\tau\right)
\end{aligned}
\tag{2.4}
$$

Note that, if one neglects the elasticity of the cylinder and the base and determines the pressure in the region of contact from the solution of Eq. (1.9) with boundary conditions (1.5), we shall have

$$
P(\xi)=\frac{D}{K}\left[\operatorname{cth}\frac{1}{\mathrm{De}}+\xi-\frac{\exp(\xi/\mathrm{De})}{\operatorname{sh}(1/\mathrm{De})}\right] \quad \left(\mathrm{De}=\frac{C}{KD}=\frac{2\eta_n V}{\lambda_n(a+b)}\right) \tag{2.5}
$$

where De is the Deborah number.

3. ANALYSIS OF THE SHEAR STRESSES IN THE CONTACT AREA

If the normal contact pressures are known, the shear stresses in the contact area can be obtained from relations (1.1), (1.3), (1.6), (1.8) and (1.10). The following integral equation for determining the function $\tau(x)$ holds in the adhesion zone A

$$
-\frac{h}{V\eta_\tau}\tau(x)+\frac{h}{\lambda_\tau}\frac{d\tau(x)}{dx}-\frac{2}{\pi E^*}\int_{-a}^{b}\frac{\tau(s)}{x-s}ds=-\delta-\frac{2\beta}{\pi E^*}p(x), \quad x\in A \tag{3.1}
$$

$$
\beta=\pi E^*[(1-2\nu_1)(1+\nu_1)/E_1-(1-2\nu_2)(1+\nu_2)/E_2]/2
$$

Using the method described above to analyse contact pressures, Eq. (3.1) can be reduced to the form

$$
\rho(\xi)=0, \quad \xi\in A_\xi \tag{3.2}
$$

where

$$
\begin{aligned}
&\rho(\xi)=\delta+\beta P(\xi)+\frac{C_1}{D}\Phi(\xi)-\int_{-1}^{1}\left[\ln|\xi-t|+\frac{K_1}{2}\operatorname{sgn}(\xi-t)\right]\Phi(t)dt \\
&\Phi(\xi)=q'(\xi), \quad q(\xi)=\frac{2}{\pi E^*}\tau\left(\frac{b-a}{2}+\frac{a+b}{2}\xi\right), \quad C_1=\frac{h\pi E^*}{2R\lambda_\tau}, \quad K_1=\frac{h\pi E^*}{2V\eta_\tau}
\end{aligned}
\tag{3.3}
$$

Moreover, in the adhesion zone A_ξ, the shear stresses satisfy the inequality $|\, q(\xi)\,| < \mu P(\xi)$, which follows from (1.4).

Relation (1.1) serves to determine the tangential stresses in the zones S where sliding occurs. Furthermore, in these zones, the shear stresses are opposite to the sliding direction, which leads to the relation

$$
\tau(x)=\mu p(x)\operatorname{sgn}\left(\frac{du_2}{dx}-\frac{du_1}{dx}+\frac{du_3}{dx}+\delta\right)
$$

Substituting (1.8) and (1.10) into this and using (3.3), we obtain

$$q(\xi)=\mu P(\xi)\operatorname{sgn}\rho(\xi),\quad \xi\in S_\xi \tag{3.4}$$

The continuity equation

$$q(\xi_i)=\mu P(\xi_i),\quad i=1,2,\dots,k \tag{3.5}$$

holds at the points ξ_i where one zone changes into another.

Equations (3.2), (3.4) and (3.5) are used to find the shear stresses in the contact area and, also, the positions and dimensions of the zones where there is adhesion and sliding. An iterative process was used for the numerical analysis of the equations obtained.

The problem of finding the shear stresses is simplified considerably by assuming that $2\lambda_\tau/E^* \ll 1$ and $\beta=0$. In this case, Eqs (3.2) and (3.4) reduce to the following equations

$$q(\xi)-\mathrm{De}_\tau\frac{dq}{d\xi}=B,\quad \xi\in A_\xi \tag{3.6}$$

$$q(\xi)=\mu P(\xi)\operatorname{sgn}\left[-q(\xi)+\mathrm{De}_\tau\frac{dq}{d\xi}+B\right],\quad \xi\in S_\xi \tag{3.7}$$

$$\left(\mathrm{De}_\tau=\frac{2V\eta_\tau}{\lambda_\tau(a+b)},\quad B=\frac{2\delta V\eta_\tau}{h\pi E^*}\right)$$

Analysis of Eqs (3.6) and (3.7) shows that, in the case being considered, the contact area is subdivided into two zone (adhesion and sliding) or three zones (sliding, adhesion and sliding).

In the case where there are two zones the shear stresses in the contact area are defined by the expression

$$q(\xi)=\begin{cases}\mu P(\xi), & \xi\in(-1,\xi_1)\\ B(1-\kappa(\xi-1)), & \xi\in(\xi_1,1)\end{cases}\qquad \left(\kappa(x)=\exp\left(\frac{x}{\mathrm{De}_\tau}\right)\right) \tag{3.8}$$

and the transition point ξ_1 is found from the relation

$$B(1-\kappa(\xi_1-1))=\mu P(\xi_1) \tag{3.9}$$

This case occurs if

$$\mathrm{De}_\tau\,\mu P'(1-0)+B\leqslant 0 \tag{3.10}$$

In the opposite case, that is, when there are three zones, we have the following expression for determining the shear stresses

$$q(\xi)=\begin{cases}\mu P(\xi), & \xi\in(-1,\xi_1)\cup(\xi_2,1)\\ B+[\mu P(\xi_2)-B]\kappa(\xi-\xi_2), & \xi\in(\xi_1,\xi_2)\end{cases} \tag{3.11}$$

where ξ_1 and ξ_2 are the solution of the equations

$$B+[\mu P(\xi_2)-B]\kappa(\xi_1-\xi_2)=\mu P(\xi_1) \tag{3.12}$$

$$\mu P(\xi_2)-B-\mu\,\mathrm{De}_\tau P'(\xi_2)=0 \tag{3.13}$$

When there is no visco-elastic layer, only two zones (adhesion and sliding) exist in the contact area when an elastic cylinder rolls over a base made of the same material ($\beta=0$).

4. ANALYSIS OF THE ROLLING RESISTANCE

A rolling cylinder is acted upon by a normal active load w and a tangential active load Q, a moment m and, also, the reactions of the base w_1 and Q_1 which arise as the result of the action of the normal and shear stresses in the contact area (Fig. 1). The equation

$$m-\int_{-a}^{b} xp(x)dx+Q_1R=0$$

follows from the condition of equilibrium of the forces and moments.

Using the notation introduced in (2.1) and (3.3), we obtain the following expressions for the dimensionless magnitudes of the resistive force and moment of rolling friction

$$T=\frac{2Q}{\pi RE^*}=-D\int_{-1}^{1}\xi\Phi(\xi)d\xi,\quad M=\frac{2m}{\pi E^*R^2}=-\frac{D^2}{2}\int_{-1}^{1}(\xi+L)^2F(\xi)d\xi-T \tag{4.1}$$

The first (or second) equation of (4.1) can also be used to find the magnitude of the relative sliding δ (1.3), if the value of the tangential force T (or the moment M) is known.

When $T=\mu W$, sliding occurs over the whole contact area. The case when $T=0$ corresponds to pure rolling. The coefficient of rolling friction is found from the relation

$$\mu_R=M/W \tag{4.2}$$

where the values of M and W are found using the second formula in (4.1) and formula (2.3) respectively.

5. RESULTS OF CALCULATIONS

Graphs of the contact pressure function $p(\xi)/p_0$ (p_0 is the Hertz maximum contact pressure) constructed for $C=0.1, D=0.1$ and different values of the parameter K are shown in Fig. 2. The solid curves correspond to the general case of the contact interaction of elastic bodies when there is a visco-elastic layer between them, and the dashed curves were constructed using formula (2.5) in the case when the elastic properties of the indentor and the base are neglected. The results show that, as the velocity V of displacement of the indentor decreases, that is, as the parameter K increases (see 2.1)), the curve representing the pressure distribution under the punch becomes more asymmetric. For a fixed size of the contact area and specified visco-elastic characteristics of the layer, the contact pressures and their maximum values depend very much on the elastic properties of the indentor and the base at translational velocities corresponding to small values of K. However, when the velocity decreases ($K=10$), the difference between the pressure distribution in the two cases becomes negligibly small. Hence, a visco-elastic layer mainly influences the contact pressure distribution at low velocities of motion.

Graphs of the dimensionless length D/D_0 of the contact area (D_0 is the dimensionless length of the contact area in the case of the Hertz formulation, $D_0=(2W)^{1/2}$), its displacement L and the maximum penetration Δ_{max} (2.4) of the cylinder into the visco-elastic layer against the parameter C/K when $W=0.001$ for $C=1$ (curve 1) and $C=0.1$ (curve 2) are plotted in Fig. 3. The parameter $C/K=\eta_nV/(\lambda_nR)$ depends on the relaxation time η_n/λ_n and the velocity V. It is seen that, as the parameter C/K increases, the length of the contact area decreases and tends to a constant value ($D=1.49\,D_0$ and $D=2.71\,D_0$ when $C=0.1$ and $C=1$, respectively). For small values of the parameter C/K, the length of the contact area increases considerably, especially as the parameter C increases (Fig. 3a). We note that the parameter C depends on the thickness of the layer and the relative elastic characteristics of the layer and the base. As the relaxation time becomes shorter and the translational velocity of the indentor decreases, that is, as the parameter C/K decreases, there is an increase in the displacement L of the contact area (Fig. 3b) and the maximum penetration Δ_{max} of the cylinder into the visco-elastic layer (Fig. 3b), which is due to the manifestation of the visco-elastic properties of the intermediate layer. In the case of an increase in the parameter C/K, the displacement of the contact area becomes negligibly small for all values of the parameter C.

The results of the calculations of the shear stresses in the contact area between a rolling cylinder and a base with a surface layer on it, based on an analysis of Eqs (3.2), (3.4) and (3.5), are shown in Fig. 4. The properties of the surface visco-elastic layer in this analysis are described by the parameter $\theta=\eta_\tau\lambda_n/(\eta_n\lambda_\tau)$ which is the ratio of the relation times in the tangential and normal directions ($\theta=C_1K/(CK_1)$) and, also, by the dimensionless parameter C_1 (3.3), which depends on the relative thickness of the layer and the relative elastic characteristics of the layer and the base. Plots of the distributions of the tangential contact stresses were constructed for the following values of the dimensionless parameters $C=0.1, K=1, W=0.01, \mu=0.1, C_1=0.1$. Curve 1 corresponds to $T=0.6\mu W$, $\theta=0.1$, $\beta=-0.4$, curve 2 to $T=0.8\mu W$, $\theta=1$, $\beta=-0.4$, curve 3 to $T=0.8\mu W$, $\theta=0.1$ $\beta=-0.4$, curve 4 to $T=0.8\mu W$, $\theta=0.1$, $\beta=0.4$ and curve 5 to $T=\mu W$.

The results show that, as the parameter θ increases, there is an increase in the values of the maximum shear stresses in the contact area and a decrease in the size of the adhesion zone. Furthermore, it was established that,

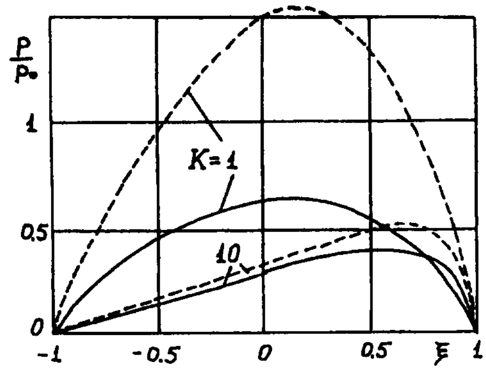

Fig. 2.

as the value of the tangential force T becomes smaller, the contact passes from a completely sliding contact (curve 5) to the three-zone and, then, to the two-zone cases. The same results were obtained qualitatively in calculations using formulae (3.8)–(3.13) for the special case of identical elastic characteristics of the cylinder and the base ($\beta = 0$) and $\lambda_v/E^* \ll 1$. With the same layer characteristics ($C_1 = 0.1$ and $\theta = 0.1$), a change in the relative elastic characteristics of the cylinder and the base from $\beta = -0.4$ (curve 3) to $\beta = 0.4$ (curve 4) leads to a transition from a three-zone contact to a two-zone contact.

Graphs of the coefficient of rolling friction, calculated using formula (4.2), against the dimensionless parameter C/K for $W = 0.001$ and $T = 0$ are also shown in Fig. 4. The coefficient of rolling friction for the model of a visco-elastic layer under consideration decreases monotonically as the parameter C/K increases and $\mu_R \to 0$ as $C/K \to +\infty$.

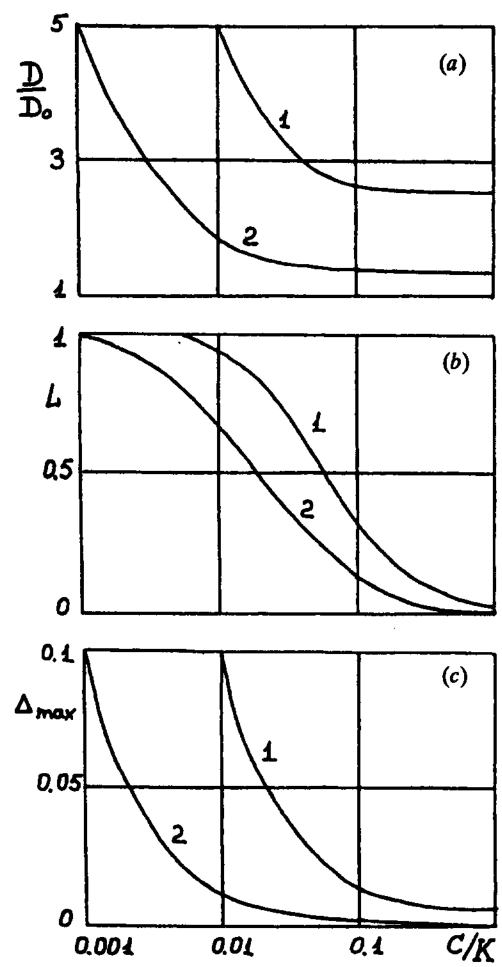

Fig. 3.

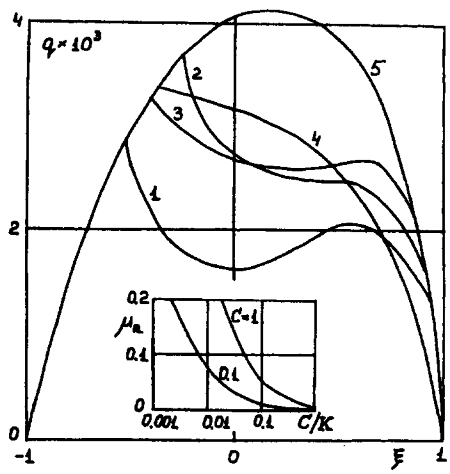

Fig. 4.

6. CONCLUSIONS

The following conclusions can be drawn from the results of the analytic and numerical analysis of the solution of the contact problem on the rolling (sliding) of an elastic cylinder along the boundary of a visco-elastic layer lying on an elastic support.

1. The normal and shear stresses in the contact area can be described using the dimensionless parameters $C, K, \theta, C_1, \beta, \mu, W, T$.

2. The ranges of variation of the parameters C, K, θ and C_1, describing the relative properties of the visco-elastic layer, for which the thin surface layer has a substantial effect on the contact characteristics have been established.

3. As the sliding velocity increases, the effect of the properties of the surface layer on the contact characteristics decreases.

4. The elastic characteristics of the indentor and the base turn out to have a relatively substantial effect on the distribution of the normal and shear stresses in the contact area, which means that they should be taken into account when formulating the problem.

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